

# Generalised Parton Distributions of the Pion on the Transverse Lattice\*

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The quark generalised parton distributions of the pion are calculated from lightcone wavefunctions in transverse lattice gauge theory at large  $N_c$ . The pion effective size is found to decrease with increasing momentum transfer. An analytic ansatz, consistent with finite boundstate lightcone energy conditions, is given for the lightcone momentum dependence of the wavefunctions. This leads to simple, universal predictions for the behaviour of the distributions near the endpoints, complementing numerical DLCQ data.

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## I. INTRODUCTION

The generalised parton distribution of the pion has been studied in a number of different models [1]. Following earlier work [2], van de Sande and the author [3] recently computed numerically the lightcone wavefunctions of the pion in transverse lattice gauge theory at large  $N_c$  [4, 5] (see also [6]). In this talk, these wavefunctions will be applied to deduce the generalised quark parton distributions in both transverse momentum and position space [7]. We also present an analytic ansatz for the lightcone momentum dependence of the wavefunctions that leads to simple, universal predictions for endpoint behaviour of the distributions, complementing the numerical data.

In transverse lattice gauge theory in  $3+1$  dimensions, two coordinates  $x^\alpha$ ,  $\alpha \in \{0, 3\}$ , are continuous while two directions  $\mathbf{x} = \{x^1, x^2\}$  are discrete. There are longitudinal continuum gauge potentials  $A_0, A_3$ , transverse flux link fields  $M_1, M_2$ , and fermions  $\Psi$ . For transverse lattice spacings  $a$  of order the hadron size, the strategy is to perform a colour-dielectric expansion [8] of the most general lightcone Hamiltonian, renormalisable with respect to the continuum coordinates, in powers of  $\Psi, M_1, M_2$  after eliminating  $A_0, A_3$  by lightcone gauge fixing. Provided the fields  $\Psi, M_1, M_2$  are chosen sufficiently heavy, one can truncate this expansion to study the low lying hadron boundstates dominated by just a few particles of these fields. The remaining free couplings in the effective Hamiltonian are fixed by optimizing symmetries broken by the regulator and, if necessary, phenomenology.

We take the pion wavefunctions from Ref. [3], which contains details of the construction of the lightcone Hamiltonian and Fock space, the renormalisation and the determination of various residual couplings appearing in the lightcone hamiltonian. The couplings were constrained by optimizing covariance of low-lying meson and glueball wavefunctions, rotational invariance of the heavy source potential, and fitting two phenomenological parameters in the meson sector, conveniently taken to be  $f_\pi$  and the  $\rho$  meson mass. In addition, two fundamental scales are taken as given, the  $\pi$  meson mass and the string tension  $\sqrt{\sigma}$ .

A typical large- $N_c$  meson eigenstate state consists of linear combinations of gauge invariant basis states formed from quarks separated by a chain of transverse links  $\overline{\Psi}MM \cdots M\Psi$ . If we label the transverse orientation of link fields by indices  $\lambda_j \in \{\pm 1, \pm 2\}$ , the partonic decomposition for a meson of momentum  $P = (P^-, P^+, \mathbf{P})$  when the transverse momentum  $\mathbf{P} = 0$  can be written

$$|P^+, \mathbf{P} = \mathbf{0}\rangle = \sum_{n=2}^{\infty} \int [dx]_n \sum_{\mathbf{z}_1, \mathbf{z}_n, h, h'} \left[ \sum_{\lambda_j} \right]_n \psi_n(x_i, h, h', \lambda_j) \times |(x_1, h, \mathbf{z}_1); (x_2, \lambda_1, \mathbf{z}_2)); \dots; (x_{n-1}, \lambda_{n-2}, \mathbf{z}_{n-1}); (x_n, h', \mathbf{z}_n)\rangle , \quad (1)$$

where  $h, h'$  denote quark and anti-quark helicities respectively,  $\mathbf{z}_i$  is the transverse position and  $x_i$  the fraction of  $P^+$  carried by the  $i^{\text{th}}$  parton. We have

$$\int [dx]_n = \int dx_1 \cdots dx_n \delta \left( \sum_{i=1}^n x_i - 1 \right) \quad (2)$$

and  $\left[ \sum_{\lambda_j} \right]_n$  indicates that the sum over orientations of links must form an unbroken chain on the transverse lattice between quark and anti-quark. We define a set of hadron states boosted to general transverse momentum  $\mathbf{P}$  by applying the Poincaré generators  $\mathbf{M}^+ = (M^{+1}, M^{+2})$ .

This gives for each parton Fock state

$$\begin{aligned} & \exp \left[ -i \mathbf{M}^+ \cdot \mathbf{P} / P^+ \right] |(x_1, h, \mathbf{z}_1); \dots; (x_n, h', \mathbf{z}_n) \rangle \\ &= \exp [i \mathbf{P} \cdot \mathbf{c}] |(x_1, h, \mathbf{z}_1); \dots; (x_n, h', \mathbf{z}_n) \rangle , \end{aligned} \quad (3)$$

$$\mathbf{c} = (c^1, c^2) = \sum_{i=1}^n x_i \mathbf{z}_i . \quad (4)$$

The lightcone wavefunctions  $\psi_n(x_i, h, h', \lambda_j)$  computed in Ref.[3] were subject to a number of cutoffs. Firstly, the transverse lattice spacing is held fixed at  $a \approx 300 \text{ MeV}^{-1}$ , representing the renormalisation scale. Thus, it would be inappropriate to use momentum transfers  $Q$  much larger than this. A cutoff  $n \leq 5$  was used. This translates into a maximum transverse size for the meson of 3 links, or about 2 fm. The effect is to reduce slightly the charge radius of the pion from its experimental value, as was shown in the form factor  $F(Q^2)$  calculated in Ref.[7]. Finally, calculations were performed in DLCQ, which discretizes the momentum fractions  $x_i$  in units of a small cutoff  $1/K$ , to create a finite-dimensional Fock space. The largest basis studied was  $K = 20$ , which is of dimension 82,470. Observables tend not to change much when extrapolated from  $K = 20$  to  $K = \infty$ , except close to the edges of phase space. We will give an analytic approach to these edges in section IV

## II. NUMERICAL GENERALISED PARTON DISTRIBUTION

We follow the conventions of Diehl [9], defining the generalised distributions in lightcone gauge as

$$\mathcal{H}(\bar{x}, \xi, Q^2) = \frac{1}{\sqrt{1 - \xi^2}} \int_{-\infty}^{+\infty} \frac{dz^-}{4\pi} e^{i\bar{x}\bar{P}^+ z^-} \langle P_{\text{in}} | \bar{\Psi}(-z^-/2) \gamma^+ \Psi(z^-/2) | P_{\text{out}} \rangle , \quad (5)$$

$$Q = P_{\text{in}} - P_{\text{out}} \quad (6)$$

$$\xi = \frac{(P_{\text{in}} - P_{\text{out}})^+}{2\bar{P}^+} \quad (7)$$

$$\bar{P}^+ = \frac{(P_{\text{in}} + P_{\text{out}})^+}{2} \quad (8)$$

Unfortunately, we are not able to explore the region  $\bar{x} < \xi$  with the strict large- $N_c$  wavefunctions. That region is governed by their  $1/N_c$  corrections, since it depends upon quark pair production. Even at  $\bar{x} > \xi$ , in DLCQ one needs to form overlaps of wavefunctions with different  $K$  in fixed ratio for fixed  $\xi$ . Existing data do not cover enough values of  $K$ . Thus, we concentrate on  $\xi = 0$  for the DLCQ analysis.

The  $K = 20$  data are displayed in fig.1 for two extreme (!) values of  $Q$  at  $\xi = 0$ .  $\mathcal{H}(x, 0, 0)$  is the conventional structure function [3], which exhibits rising Regge behaviour at small  $x$  characteristic of a system with gluonic degrees of freedom. It falls to zero precipitously near  $x \rightarrow 1$ . This may be an indication that the chiral limit is near (the pion mass was fit to 140 MeV), since chirally symmetric quark models [10] produce a flat distribution. As purely transverse  $Q$  is increased, the distribution is seen to deplete at small  $x$  but is  $Q$ -independent as  $x \rightarrow 1$ .

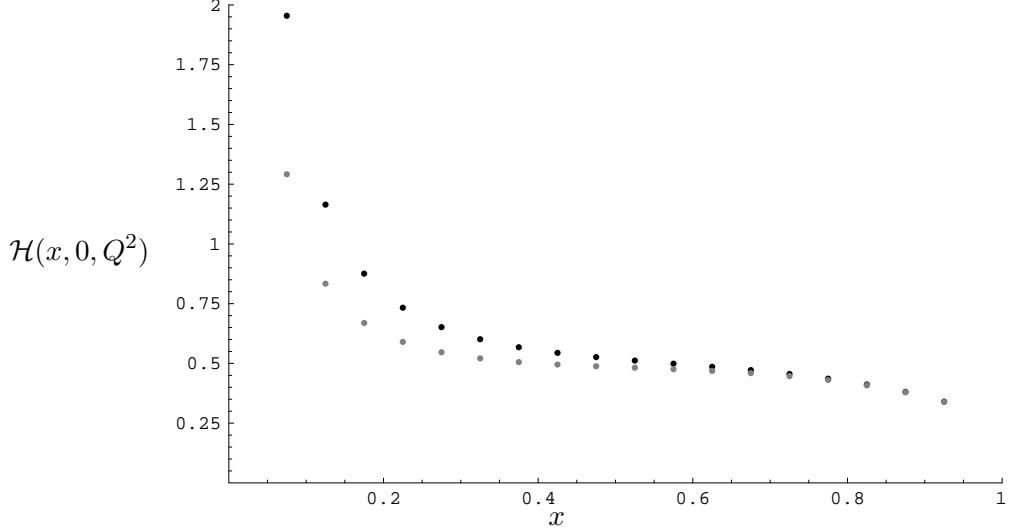


FIG. 1: Generalised quark distribution of the pion at zero skewness and DLCQ cutoff  $K = 20$ : black points are  $Q = 0$ , grey are  $Q = 500$  MeV.

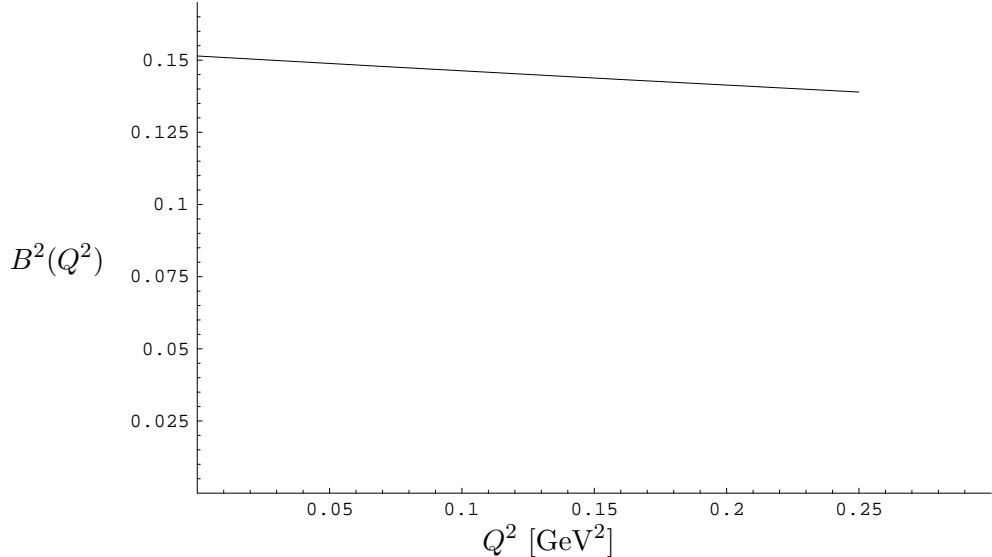


FIG. 2: Effective size of the pion as measured by Eq. (9).

A measure of the effective size of the pion is given by the average distance between the active quark and the center of momentum of the spectators [11]

$$B^2(Q^2) = -\frac{1}{F(Q^2)} \frac{\partial}{\partial Q^2} \int_0^1 dx \frac{\mathcal{H}(x, 0, Q^2)}{(1-x)^2}. \quad (9)$$

This is plotted in fig. 2. The authors of Ref. [11] claim that  $B^2(Q^2 \rightarrow \infty) \rightarrow 0$  is a condition for colour transparency. The data are consistent with the pion size reducing, but obviously it is impossible to reliably extrapolate into the large momentum transfer region.

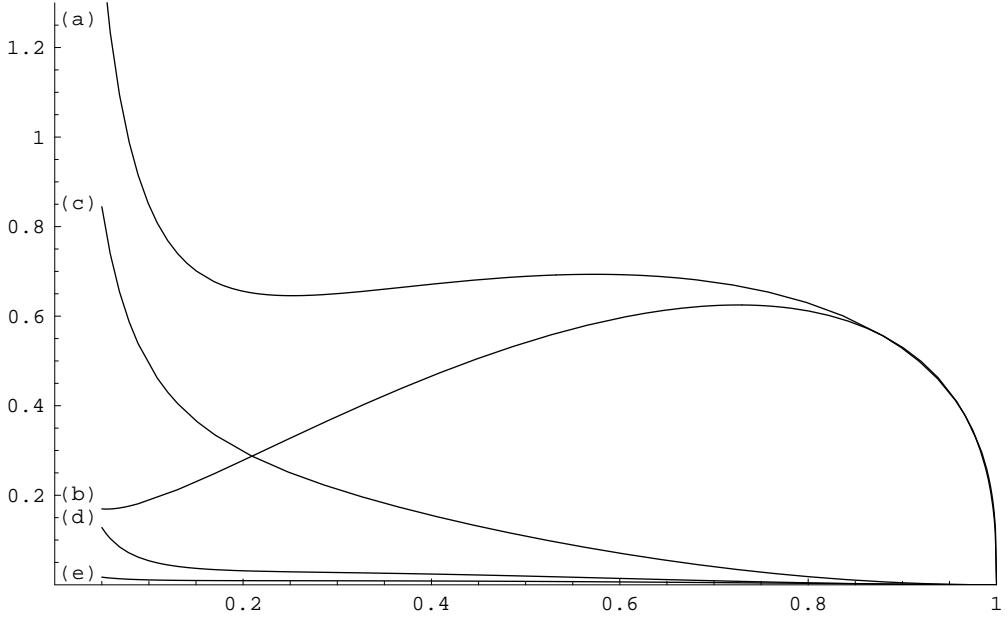


FIG. 3: Impact parameter dependent valence quark distributions of the pion. (a) The full distribution  $\mathcal{H}(x, 0, 0)$ , which sums over all impact parameters  $\mathbf{b}$ . (b)-(e) The sum of contributions at impact parameters  $\mathbf{b}$  along lattice axes such that  $b = |\mathbf{b}| = 0, a, 2a, 3a$  respectively.

### III. IMPACT PARAMETER DISTRIBUTION

The impact parameter dependent quark distribution was analysed in Ref.[7]. It is defined by [12]

$$\mathcal{I}(\bar{x}, \xi, \mathbf{b}) = \frac{1}{\sqrt{1 - \xi^2}} \langle P_{\text{out}}^+, \mathbf{b}_0^{\text{out}} | \int \frac{dz^-}{4\pi} e^{i\bar{x}\bar{P}^+ z^-} \times \bar{\Psi}(-z^-/2, \mathbf{b}) \gamma^+ \Psi(z^-/2, \mathbf{b}) | P_{\text{in}}^+, \mathbf{b}_0^{\text{in}} \rangle , \quad (10)$$

where

$$\mathbf{b}_0^{\text{out}} = -\frac{\xi}{1-\xi} \mathbf{b} , \quad \mathbf{b}_0^{\text{in}} = \frac{\xi}{1+\xi} \mathbf{b} . \quad (11)$$

$\mathbf{b}_0$  is the transverse position of the hadron, assumed spinless.  $\mathcal{I}(x, 0, \mathbf{b})$  is simply the probability of a quark carrying fraction  $x$  of the lightcone momentum  $P^+$  when at transverse position  $\mathbf{b}$  [13].

To translate results to hadron impact parameter space in the transverse directions, we integrate  $\mathbf{P}$  over the Brillouin zone to obtain a hadron state localised at transverse position  $\mathbf{0}$

$$|P^+, \mathbf{b}_0 = \mathbf{0}\rangle = \sum_{n=2}^{\infty} \int [dx]_n \sum_{\mathbf{z}_1, \mathbf{z}_n, h, h'} \left[ \sum_{\lambda_j} \right]_n \frac{a^2}{\pi^2} \frac{\sin c^1 \pi/a}{c^1} \frac{\sin c^2 \pi/a}{c^2} \times \psi_n(x_i, h, h', \lambda_j) |(x_1, h, \mathbf{z}_1); \dots; (x_n, h', \mathbf{z}_n)\rangle \quad (12)$$

Hence

$$\begin{aligned} \mathcal{I}(x, 0, \mathbf{b}) = & \sum_{n=2}^{\infty} \int [dx]_n \sum_{\mathbf{z}_1, \mathbf{z}_n, h, h'} \left[ \sum_{\lambda_j} \right]_n \delta(x_1 - x) \delta(\mathbf{z}_1 - \mathbf{b}) \times \\ & \left[ \frac{a^2 \sin c^1 \pi/a}{\pi^2} \frac{\sin c^2 \pi/a}{c_1} \frac{c_2}{c_2} \right]^2 |\psi_n(x_i, h, h', \lambda_j)|^2. \end{aligned} \quad (13)$$

Due to the lattice cutoff, this should strictly be interpreted as the probability of finding the quark within one lattice spacing of impact parameter  $\mathbf{b}$ . We show results for the first few values of  $b = |\mathbf{b}|$  sampled along lattice axes in fig. 3. In this case, the curves are the result of an extrapolation to  $K = \infty$  from data at  $K = 10, 12, 15, 20$ . Fig. 3(a) shows the full distribution  $\mathcal{H}(x, 0, 0)$ , which sums over all impact parameters  $\mathbf{b}$  (one can compare this with the  $K = 20$  version in Fig. 1). Fig. 3(b) is the contribution from  $\mathbf{b} = (0, 0)$ , Fig. 3(c) is the sum of contributions from  $\mathbf{b} = (a, 0), (0, a), (-a, 0), (0, -a)$ , etc.. This clearly shows that the valence region  $x > 0.5$  is dominated by quarks with impact parameter less than the charge radius  $\sim 2/3$  fm. It also suggests that the rapid rise at small  $x$  is dominated by larger impact parameters  $b > 2/3$  fm. This all implies a sharp fall-off of the hadron wavefunction in transverse space at a particular ( $x$ -dependent) radius.

#### IV. ANALYTIC WAVEFUNCTION ANSATZ

The boundstate problem can be cast into a set of integral equations for the wavefunctions  $\psi_n$

$$M^2 \psi_n(x_i, h, h', \lambda_j) = \int \mathcal{K} \cdot \psi_n \quad (14)$$

where, on the right hand side, the wavefunction is convoluted with a kernel  $\mathcal{K}$  [3] that exchanges lightcone momentum, creates, and annihilates partons. Typically the kernel is singular when one or more momentum fractions  $x_i$  vanish in this equation. Demanding that the boundstate mass  $M$  remains finite, one can constrain the behaviour of the wavefunctions  $\psi_n$  in these limits [14]. A thorough analysis of the endpoint behaviour of transverse lattice wavefunctions will be given in Ref. [15]. Here we will give the simplest ansatz for the first few  $\psi_n$  in a large- $N_c$  meson that has the correct endpoint behaviour when any number of  $x_i$  vanish:

$$\psi_2(x) = C_2 x^\alpha (1-x)^\alpha \quad (15)$$

$$\psi_3(x, y) = C_3 \frac{y^\beta (1-y)^\alpha}{[(x+y)(1-x)]^{\beta-\alpha+1/2}} \quad (16)$$

$$\begin{aligned} \psi_4(x, y, z) = & C_4 \frac{(yz)^\beta [(1-y)(1-z)]^\alpha [(x+y+z)(1-x)]^{\alpha+\beta-1/2}}{[(x+y)(1-x-y)]^{\beta+1/2} (y+z)^{2\beta}} \\ & \dots \end{aligned} \quad (17)$$

The exponents  $\alpha$  and  $\beta$  are independent of  $n$ , quark helicity, and the transverse shape, while the coefficients  $C_n$  are dependent on all of these factors. They are all determined by the solution to Eqs. (14). The coefficients  $C_n$  are complex in general, although overlaps of wavefunctions appearing in  $\mathcal{H}$  are real because the theory is  $PT$  invariant. For normalisability one must have  $\beta < 1/2$ . The simplest forms above could be generalised for use as

a variational basis, by distinguishing quark and link-field  $x_i$  and/or multiplying them by a complete basis of analytic functions of the  $x_i$ . In principle, one can use DLCQ data to determine the unknown parameters but, in practice, it is difficult because convergence is slow in the endpoint regions. We could, however, obtain a fairly reliable fit  $\alpha \approx 1/2$  from the data of Ref. [3], implying a quark distribution function of the form  $\sqrt{x(1-x)}$  at this low scale.

Even without determining the unknown parameters, one can deduce a number of interesting conclusions. Firstly, one notes that the diverging singularity of the wavefunctions  $\psi_n$  becomes stronger as more momentum fractions vanish. Intuition from the free lightcone kinetic energy  $\sim \sum_i m_i/x_i$ , for partons of mass  $m_i$ , would have suggested the complete opposite. Rather than simply vanishing at the edges of phase space, as indicated by the free theory, the interactions force relations between wavefunctions with different numbers of partons to cancel the divergences in Eq. (14). This was noted for continuum QCD for a single vanishing  $x_i$  in the first reference of Ref.[14], but for the transverse lattice we are able to generalise it to any number of vanishing momenta. This allows us to make a number of definite statements relevant to generalised quark parton distributions:

1. each  $\psi_n$  contributes const.  $\times (1 - \bar{x})^{2\alpha}$  to  $\mathcal{H}(\bar{x}, \xi, Q^2)$  as  $\bar{x} \rightarrow 1$ ;
2.  $\psi_n$  contributes  $(\ln \frac{1}{x})^{n-3}$ ,  $n > 2$ , to  $\mathcal{H}(x, 0, 0)$  as  $x \rightarrow 0$ ;
3. each  $\psi_n$ ,  $n > 2$ , contributes a finite constant to  $\mathcal{H}(\bar{x}, \xi, Q^2)$  as  $\bar{x} \rightarrow \xi$  from above;
4.  $\partial \mathcal{H}(x, 0, Q^2) / \partial Q^2|_{Q^2=0} \sim (1 - x)^2$  as  $x \rightarrow 1$ .

Provided the constants fall sufficiently fast with  $n$ , continuity [9] and item 3 above suggest that  $\mathcal{H}(\xi, \xi, Q^2)$  is finite.

## V. OUTLOOK

The transverse lattice has so far proven to be the only systematically improvable non-perturbative lightcone method that preserves the gauge invariance crucial to confinement. The current DLCQ transverse lattice calculations for mesons can be improved by going to higher orders of the colour dielectric expansion and testing directly for  $P$ ,  $T$ , and chiral symmetry. As well as the initial applications to the pion presented here, one could also calculate the generalised distributions of the heavier mesons. Of course, most experimental data derive from the nucleon, so it is important to develop computations in the baryon sector. These have proven awkward in the past because the numerical work does not simplify in the large- $N_c$  limit. However, many of the analytic arguments above will also be valid for baryons.

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